

# STANLEY DECOMPOSITIONS IN LOCALIZED POLYNOMIAL RINGS

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**ABSTRACT.** We introduce the concept of Stanley decompositions in the localized polynomial ring  $S_f$  where  $f$  is a product of variables, and we show that the Stanley depth does not decrease upon localization. Furthermore it is shown that for monomial ideals  $J \subset I \subset S_f$  the number of Stanley spaces in a Stanley decomposition of  $I/J$  is an invariant of  $I/J$ . For the proof of this result we introduce Hilbert series for  $\mathbb{Z}^n$ -graded  $K$ -vector spaces.

## INTRODUCTION

Let  $K$  be a field,  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ ,  $A \subset \{1, 2, \dots, n\}$  and  $f = \prod_{j \in A} x_j$ . Then  $S_f$  is a  $\mathbb{Z}^n$ -graded  $K$ -algebra whose  $K$ -basis consists of the monomials  $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  with  $a_j \in \mathbb{Z}$  and  $a_j \geq 0$  if  $j \notin A$ .

Standard algebraic operations like reduction module regular elements or restricted localizations have been considered in the papers [5] and [6]. To better understand localization of Stanley decompositions we define in this paper Stanley decompositions of  $S_f$ -modules of the type  $I/J$  where  $J \subset I \subset S_f$  are monomial ideals in  $S_f$ . To do this we first have to extend the definition of Stanley spaces as it is given for  $\mathbb{N}^n$ -graded  $K$ -algebras. Here the  $K$ -subspace of the form  $uK[Z] \subset I/J$ , where  $u$  is a monomial in  $I \setminus J$  and  $Z \subset \{x_1, \dots, x_n\} \cup \{x_j^{-1} : j \in A\}$  such that  $\{x_j, x_j^{-1}\} \not\subset Z$  for all  $j \in A$ , is called *Stanley space* if  $uK[Z]$  is a free  $K[Z]$ -module of  $I/J$ . The dimension of the Stanley space  $uK[Z]$  is defined to be  $|Z|$ . As in the classical case we define a *Stanley decomposition* of  $I/J$  as a finite direct sum of Stanley spaces  $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_i K[Z_i]$ , and set  $\text{sdepth}(\mathcal{D}) = \min\{|Z_i| : 1 \leq i \leq r\}$  and  $\text{sdepth } I/J = \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } I/J\}$ .

Our more general definition of Stanley spaces is required for  $S_f$ , because otherwise it would be impossible to obtain a Stanley decomposition of  $S_f$  when  $f \neq 1$ . Actually we show in Proposition 2.1 that  $S_f$  has a nice canonical Stanley decomposition with Stanley spaces as defined above. Indeed, we have  $S_f = \bigoplus_{L \subseteq A} f_L^{-1} K[Z_L]$ , where

$$Z_L = \{x_l^{-1} \mid l \in L\} \cup \{x_l \mid l \notin L\}$$

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and  $f_L = \prod_{l \in L} x_l$ . In this decomposition all Stanley spaces are of dimension  $n$  and the number of summands is  $2^{|A|}$ . One may ask if there exist Stanley decompositions of  $S_f$  with less summands. We will show that this is not possible. Indeed, as a generalization of a result in [3] we show in Theorem 6.5 that number of Stanley spaces of maximal dimension in a Stanley decomposition of an  $S_f$ -module of the form  $I/J$  is independent of the particular Stanley decomposition of  $I/J$ . In order to prove this fact we introduce in Section 6 a modified Hilbert function for  $\mathbb{Z}^n$ -graded finitely generated  $K$ -vector spaces  $M$  with the property that  $\dim_K M_a < \infty$  for all  $a \in \mathbb{Z}^n$ . For such modules we set  $H(M, d) = \sum_{a \in \mathbb{Z}^n, |a|=d} \dim_K M_a$  and call the formal power series  $H_M(t) = \sum_{d \geq 0} H(M, d)t^d$  the Hilbert series of  $M$ . Here we use the notation  $|a| = \sum_{i=1}^n |a_i|$  for  $a = (a_1, a_2, \dots, a_n)$ . Our Hilbert series does not as well behave as the usual Hilbert series with respect to shifts. Nevertheless Proposition 6.2 implies that for any monomial  $u$  in  $S_f$  one has  $H_{uS_f}(t) = t^{|\deg(u')|} (1+t)^{|A|} / (1-t)^n$ , where  $u'$  is obtained from  $u$  by removing the unit factors in  $u$ . Quite generally we show that, just as for the usual Hilbert series, one has  $H_{I/J}(t) = P(t)/(1-t)^d$  where  $d$  is the Krull dimension of  $I/J$  and  $P(1) > 0$ .

Any monomial ideal  $I \subset S_f$  is obtained by localization from a monomial ideal in  $S$ . Therefore, for  $J \subset I \subset S$  it is natural to compare the Stanley depth of  $I/J$  with that of  $(I/J)_f$ . Theorem 3.1 states that  $\text{sdepth } I/J \leq \text{sdepth } (I/J)_f$ , and we give examples that show that this inequality can be strict. In Theorem 5.1 we prove a similar result for the so-called fdepth which is define via filtrations. Theorem 5.1 is related to a theorem proved by the first author in [4], where it is shown that  $\text{sdepth } S'/I_{x_1} \cap S' \geq \text{sdepth } S/I - 1$ , where  $S' = K[x_2, \dots, x_n]$ .

Let  $J \subset I \subset S$  be monomial ideals, and consider the polynomial ring  $S[t]$  over  $S$ . Herzog, Vladioiu and Zheng in [2] proved that  $\text{sdepth } (I/J)[t] = \text{sdepth } (I/J) + 1$ . In Theorem 4.1, we see that a similar relation holds between the Stanley depth of  $I/J$  and  $(I/J)[t, t^{-1}]$  for monomial ideals  $J \subset I \subset S_f$ . It implies that Stanley depth of  $I/J$ , where  $J \subset I \subset S_f$  monomial ideals, also if  $f \neq 0$  can be computed.

## 1. MONOMIAL IDEALS IN $S_f$

Let  $K$  be a field and  $S = K[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  variables over  $K$ . We fix a subset  $A \subset \{1, \dots, n\}$  and set

$$f = \prod_{j \in A} x_j.$$

As usual, the localization of  $S$  with respect to multiplicative set  $\{1, f, f^2, \dots\}$  is denoted by  $S_f$ . We note that  $S_f = K[x_1, x_2, \dots, x_n, x_j^{-1} : j \in A]$ .

The monomials in  $S_f$  are the element  $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$  with  $a_j \in \mathbb{Z}$  if  $j \in A$  and  $a_j \in \mathbb{N}$  if  $j \notin A$ . The set  $\text{Mon}(S_f)$  of monomials is a  $K$ -basis of  $S_f$ . An ideal  $I \subset S_f$  is called a *monomial ideal* if it is generated by monomials. A minimal set of monomial generators of  $I$  is uniquely determined, and is denoted  $G(I)$ .

Any element  $g \in S_f$  is a unique  $K$ -linear combination of monomials.

$$g = \sum a_u u, \quad \text{with } a_u \in K \quad \text{and} \quad u \in \text{Mon}(S_f).$$

We call the set

$$\text{Supp}(g) = \{u \in \text{Mon}(S_f) : a_u \neq 0\}$$

the support of  $g$ .

For monomial  $u = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ , we set  $\text{supp}(u) = \{i \in [n] : a_i \neq 0\}$ ,  $\text{supp}_+(u) = \{i \in [n] : a_i > 0\}$  and  $\text{supp}_-(u) := \{i \in [n] : a_i < 0\}$ , and call these sets *support*, *positive support*, and *negative support* of  $u$ , respectively.

Just as for monomial ideals in the polynomial ring one shows that the set of monomials  $\text{Mon}(I)$  belonging to  $I$  is a  $K$ -basis of  $I$ . We obviously have

**Proposition 1.1.** *Let  $I \subset S_f$  be an ideal. The following are equivalent.*

- (a)  $I \subset S_f$  is a monomial ideal.
- (b) There exists a monomial ideal  $I' \subset S$  such that  $I = I'S_f$ .

*If the equivalent conditions hold, then  $I'$  can be chosen such that  $\text{supp}(u) \subset [n] \setminus A$  for all  $u \in G(I')$ . The monomial ideal  $I'$  satisfying this extra condition is uniquely determined. Indeed,  $I' = I \cap S$ .*

Let  $J \subset I \subset S_f$  be monomial ideals. Observe that the residue classes of the monomials belonging to  $I \setminus J$  form a  $K$ -basis of the residue class  $I/J$ . Therefore we may identify the classes of  $I/J$  with the monomials in  $I \setminus J$ .

## 2. STANLEY DECOMPOSITION OF $I/J$

Let  $J \subset I \subset S_f$  be monomial ideals and  $f = \prod_{j \in A} x_j$ . A *Stanley space* of  $I/J$  is a free  $K[Z]$ -submodule  $uK[Z]$  of  $I/J$ , where  $u$  is a monomial of  $I/J$  and

$$Z \subset \{x_1, \dots, x_n\} \cup \{x_j^{-1} : j \in A\},$$

satisfying the condition that  $\{x_j, x_j^{-1}\} \not\subset Z$  for all  $j \in A$ .

A *Stanley decomposition* of  $I/J$  is a finite direct sum of Stanley spaces

$$\mathcal{D} : I/J = \bigoplus_{i=1}^r u_i K[Z_i].$$

We set

$$\text{sdepth } \mathcal{D} = \min\{|Z_i| : 1 \leq i \leq r\},$$

and

$$\text{sdepth}(I/J) = \max\{\text{sdepth } \mathcal{D} : \mathcal{D} \text{ is a Stanley decomposition of } I/J\}.$$

This number is called the *Stanley depth* of  $I/J$ .

**Proposition 2.1.** *The ring  $S_f$  admits the following canonical Stanley decomposition*

$$\mathcal{D} : S_f = \bigoplus_{L \subseteq A} f_L^{-1} K[Z_L],$$

where  $Z_L = \{x_l^{-1} \mid l \in L\} \cup \{x_l \mid l \notin L\}$  and  $f_L = \prod_{l \in L} x_l$ .

*Proof.* Consider a monomial  $h = x_1^{a_1} \cdots x_n^{a_n}$  of  $S_f$ . We choose

$$L = \{1 \leq l \leq n : a_l < 0\} \subset A.$$

Then

$$h = f_L^{-1}(x_1^{a_1} \cdots x_n^{a_n} f_L) = f_L^{-1}x_1^{b_1} \cdots x_n^{b_n},$$

where  $b_l = a_l \in \mathbb{N}$  if  $l \notin L$  and  $b_l = a_l + 1 \in \mathbb{Z}$  if  $l \in L$ , and hence  $h \in f_L^{-1}K[Z_L]$ . This shows that  $S_f = \sum_{L \subset A} f_L K[Z_L]$ .

To show that this sum is direct, it is enough to prove any two distinct Stanley spaces in the decomposition of  $I/J$  have no monomial in common, since the multigraded components of  $I/J$  are of dimension  $\leq 1$ .

Let  $L, L' \subset A$  with  $L \neq L'$ . Then  $f_L^{-1} \neq f_{L'}^{-1}$ . Suppose that there is a monomial  $u \in I/J$  such that

$$u \in f_L^{-1}K[Z_L] \cap f_{L'}^{-1}K[Z_{L'}],$$

that is

$$u = f_L^{-1}v = f_{L'}^{-1}v'$$

for some monomials  $v \in K[Z_L], v' \in K[Z_{L'}]$ . Since is a contradiction, because

$$L = \text{supp}_-(u) = L',$$

as follows from the above equation.  $\square$

**Example 2.2.** Let  $S = K[x, y, z]$  and  $f = yz$ . Then according to Lemma 2.1,  $S_f = K[x, y, z] \oplus y^{-1}K[x, y^{-1}, z] \oplus z^{-1}K[x, y, z^{-1}] \oplus y^{-1}z^{-1}K[x, y^{-1}, z^{-1}]$  is a Stanley decomposition of  $S_f$ .

As a consequence of Proposition 2.1 we have

**Corollary 2.3.**  $\text{sdepth}(S_f) = n$ .

### 3. LOCALIZATION OF STANLEY DECOMPOSITIONS

Let  $J \subset I \subset S$  be monomial ideals and we set  $f = \prod_{j \in A} x_j$ . The next result shows how a Stanley decomposition of  $I/J$  localizes.

**Theorem 3.1.** Let  $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_i K[Z_i]$  be a Stanley decomposition of  $I/J$ . Then  $\mathcal{D}_f : (I/J)_f = \bigoplus_{i \in C} (\bigoplus_{L \subset A} u_i f_L^{-1} K[Z_i^L])$  is a Stanley decomposition of  $(I/J)_f$ , where  $C = \{i : Z_A \subset Z_i\}$  and  $Z_i^L = \{x_l^{-1} \mid l \in L, x_l \in Z_i\} \cup \{x_l \mid l \notin L, x_l \in Z_i\}$ . In particular, we have

$$\text{sdepth } I/J \leq \text{sdepth } (I/J)_f.$$

*Proof.* Let  $u \in (I/J)_f$  be a nonzero monomial. We claim that

$$u \in \sum_{i \in C} (\sum_{L \subset A} u_i f_L^{-1} K[Z_i^L]).$$

Since  $u \in (I/J)_f$  is nonzero it follows that  $u f^a \in I \setminus J$  for all  $a \gg 0$ . Hence there exists an integer  $i$  such that  $u f^a \in u_i K[Z_i]$  for infinitely many  $a > 0$ . This

implies that  $Z_A \subset Z_i$ , that is,  $i \in C$ . So,  $uf^a \in u_i K[Z_i]$ , that is,  $uf^a = u_i v$  where  $v = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(Z_i)$ . Then  $u = u_i v f^{-a} = u_i x_1^{b_1} \cdots x_n^{b_n}$ , where

$$b_i = \begin{cases} a_i - a, & \text{if } i \in A \\ a_i, & \text{otherwise.} \end{cases}$$

Say  $L = \text{supp}_-(x_1^{b_1} \cdots x_n^{b_n})$ . Note that  $L \subset A$ . Hence  $u \in u_i f_L^{-1} K[Z_i^L]$ .

In order to prove other inclusion, consider a monomial  $w \in \sum_{i \in C} (\sum_{L \subset A} u_i K[Z_i^L])$ . This implies that there exists a  $i \in C$  and  $L \subset A$  such that  $w \in u_i K[Z_i^L]$ . Since  $Z_A \subset Z_i$ , we have  $w f^a \in u_i K[Z_i]$  for  $a \gg 0$ . It follows that  $w f^a \in I$ , so  $w \in I_f$ . On the other hand,  $w \notin J_f$ . Otherwise  $w f^a \in J$  for  $a \gg 0$ , which is a contradiction, since  $w f^a \in u_i K[Z_i]$  for  $a \gg 0$ .

Now we prove that the sum is direct. Let  $L, H \subset A$  and

$$u \in u_i f_L^{-1} K[Z_i^L] \cap u_j f_H^{-1} K[Z_j^H]$$

be a monomial such that  $i \neq j$  or  $L \neq H$ . Since  $Z_A \subset Z_i, Z_j$ , we have for  $a \gg 0$  that

$$u f^a \in u_i K[Z_i] \cap u_j K[Z_j].$$

If  $i \neq j$ , then  $u f^a = 0$  in  $I/J$  implies  $u = 0$  in  $(I/J)_f$ . If  $i = j$  and  $L \neq H$  then  $u = u_i f_L^{-1} v = u_i f_H^{-1} v'$  for some monomials  $v \in K[Z_i^L]$  and  $v' \in K[Z_i^H]$ . Since  $\text{supp}_-(u_i f_L^{-1} v) = L$  and  $\text{supp}_-(u_i f_H^{-1} v') = H$ , so  $\text{supp}_-(u) = L = H$ , which is a contradiction.  $\square$

The next examples show that in the inequality on Theorem 3.1 we may have equality or strict inequality.

**Example 3.2.** Let  $J = (xy, yz) \subset I = (y) \subset S = K[x, y, z]$  be ideals,  $\mathcal{D} : I/J = yK[y]$  is a Stanley decomposition of  $I/J$ . Thus  $\text{sdepth } \mathcal{D} = 1$ . Localizing with the monomial  $f = y$ , we obtain that  $\mathcal{D}_f : (I/J)_f = yK[y] \oplus K[y^{-1}]$  is a Stanley decomposition of  $(I/J)_f$  and  $\text{sdepth } \mathcal{D}_f = 1$ .

**Example 3.3.** Let  $J = (x^3y, x^2y^2) \subset I = (x^2, xy) \subset S = K[x, y]$  be ideals,  $\mathcal{D} : I/J = xyK[y] \oplus x^2K[x] \oplus x^2yK$  is a Stanley decomposition of  $I/J$ . Thus  $\text{sdepth } \mathcal{D} = 0$ . Localizing with the monomial  $f = x$ ,  $\mathcal{D}_f : (I/J)_f = x^2K[x] \oplus xK[x^{-1}]$  is a Stanley decomposition of  $(I/J)_f$  and  $\text{sdepth } \mathcal{D}_f = 1$ .

The following example shows that a Stanley decomposition of  $I/J$  which gives the Stanley depth of  $I/J$  may localize to a Stanley decomposition whose Stanley depth is smaller than the Stanley depth of the localization of  $I/J$ .

**Example 3.4.** Let  $I = (x, y, z) \subset S = K[x, y, z]$  be the graded maximal ideal of  $S$ .  $\mathcal{D} : I = xK[x, y] \oplus yK[y, z] \oplus zK[x, z] \oplus xyzK[x, y, z]$  is a Stanley decomposition of  $I$ . Thus  $\text{sdepth } \mathcal{D} = 2$  which is also the Stanley depth of  $I$ . Localizing with the monomial  $f = x$ , we get the Stanley decomposition  $\mathcal{D}_f$  of  $I_f$  which is

$$xK[x, y] \oplus K[x^{-1}, y] \oplus zK[x, z] \oplus zx^{-1}K[x^{-1}, z] \oplus xyzK[x, y, z] \oplus yzK[x^{-1}, y, z].$$

Thus  $\text{sdepth } \mathcal{D}_f = 2$ . However  $I_f = K[x^{\pm 1}, y, z]$ , and hence  $\text{sdepth } I_f = 3$ .

#### 4. STANLEY DECOMPOSITIONS AND POLYNOMIAL EXTENSIONS

Herzog et al. in [2, Lemma 3.6] proved that the Stanley depth of the module increases by one in a polynomial ring extension by one variable. We generalize this result to localized rings.

**Theorem 4.1.** *Let  $J \subset I \subset S_f$  be monomial ideals. Then*

$$\text{sdepth}(I/J)[t] = \text{sdepth}(I/J)[t, t^{-1}] = \text{sdepth } I/J + 1.$$

*Proof.* Let  $\mathcal{D} : I/J = \bigoplus_{i=1}^r v_i K[Z_i]$  be a Stanley decomposition of  $I/J$ . Then we obtain the Stanley decompositions

$$(I/J)[t] = \bigoplus_{i=1}^r v_i K[Z_i][t] = \bigoplus_{i=1}^r v_i K[Z_i, t]$$

$$((I/J)[t])_t = \bigoplus_{i=1}^r (v_i K[Z_i, t] \oplus v_i t^{-1} K[Z_i, t^{-1}]).$$

This implies that

$$1 + \text{sdepth } I/J \leq \text{sdepth}(I/J)[t, t^{-1}], \text{sdepth}(I/J)[t].$$

Let  $\mathcal{D}' : (I/J)[t, t^{-1}] = \bigoplus_{i=1}^r v_i K[W_i]$  be a Stanley decomposition of  $(I/J)[t, t^{-1}]$ . Next we show that

$$I/J = \bigoplus_{i \in [r]} v_i K[W_i] \cap S_f,$$

and that each  $v_i K[W_i] \cap S_f = u_i K[W_i \setminus \{t, t^{-1}\}]$  for a suitable monomial  $u_i \in S_f$ , provided  $v_i K[W_i] \cap S_f \neq 0$ .

Since the direct sum  $\mathcal{D}'$  commutes with taking the intersection with  $S_f$  and since  $(I/J)[t, t^{-1}] \cap S_f = I/J$ , the desired decomposition of  $I/J$  follows.

Suppose  $v_i K[W_i] \cap S_f \neq 0$ . Then there exists monomials  $v \in S_f$  and  $w \in K[W_i]$  such that  $v = v_i w$ . We may assume that  $v_i$  does not contain  $t$  as a factor, because  $t^{-1}$  must be a factor of  $w$  which implies that  $t^{-1} \in W_i$ . Thus we may replace  $v_i$  by the monomial which is obtained from  $v_i$  by removing the power of  $t$  which appears in  $v_i$ . Similarly we may assume  $t^{-1}$  is not a factor of  $v_i$ . Then it follows that  $v_i K[W_i] \cap S_f$  consists of all monomials  $v_i w$  with  $w \in K[W_i]$  where neither  $t$  nor  $t^{-1}$  is a factor of  $w$ . In other words,  $w \in K[W_i \setminus \{t, t^{-1}\}]$ .

From these considerations we conclude that  $1 + \text{sdepth } I/J \geq \text{sdepth}(I/J)[t, t^{-1}]$ . In the same way one proves the inequality  $1 + \text{sdepth } I/J \geq \text{sdepth}(I/J)[t]$ . This yields the desired conclusions.  $\square$

An immediate consequence of Theorem 4.1 is the following

**Corollary 4.2.**

$$\text{sdepth}(I/J)[t_1^{\pm 1}, \dots, t_r^{\pm 1}] = \text{sdepth } I/J + r.$$

Let  $J \subset I \subset S_f$  be monomial ideals, and  $S' = K[x_i : i \notin A] \subset S$ . Then there exist monomial ideals  $J' \subset I' \subset S'$  such that  $J'S_f = J$  and  $I'S_f = I$ . We have

$$I'/J'[x_i^{\pm 1} : i \in A] = I/J.$$

Hence  $\text{sdepth } I'/J' + |A| = \text{sdepth } I/J$ , by Corollary 4.2. Since the Stanley depth of  $I'/J'$  can be computed in finite number of steps by the method given by Herzog, Vladioiu and Zheng [2], the Stanley depth of  $I/J$  can be computed as well in a finite number of steps.

## 5. FDEPTH AND LOCALIZATION

Let  $J \subset I \subset S$  be monomial ideals of  $S$ . A chain  $\mathcal{F} : J = J_0 \subset J_1 \subset \dots \subset J_r = I$  of monomial ideals is called a prime filtration of  $I/J$ , if  $J_i/J_{i-1} \cong S/P_i(-a_i)$  where  $a_i \in \mathbb{Z}^n$  and where each  $P_i$  is a monomial prime ideal. We call the set of prime ideals  $\{P_1, \dots, P_r\}$  the support of  $\mathcal{F}$  and denote it by  $\text{Supp}(\mathcal{F})$ . In [2], Herzog, Vladioiu and Zheng define  $\text{fdepth}$  of  $I/J$  as follows:

$$\text{fdepth}(\mathcal{F}) = \min\{\dim S/P : P \in \text{Supp}(\mathcal{F})\}$$

and

$$\text{fdepth } I/J = \max\{\text{fdepth } \mathcal{F} : \mathcal{F} \text{ is a prime filtration of } I/J\}.$$

These definitions can be immediately extended to monomial ideals  $J \subset I \subset S_f$ . We then get

**Theorem 5.1.**

$$\text{fdepth } I/J \leq \text{fdepth}(I/J)_f.$$

*Proof.* Let  $\mathcal{F} : J = J_0 \subset J_1 \subset \dots \subset J_r = I$  be a prime filtration of  $I/J$  with  $\text{Supp}(\mathcal{F}) = \{P_1, \dots, P_r\}$ . Then we obtain a filtration  $\mathcal{F}_f : J_f = (J_0)_f \subset (J_1)_f \subset \dots \subset (J_r)_f = I_f$  with factors  $S_f/P_i S_f$ . Thus if we drop the ideals  $(J_i)_f$  for which  $S_f/P_i S_f = 0$  we obtain a prime filtration of  $(I/J)_f$ . Since  $\dim S_f/P_i S_f = \dim S/P_i$ , whenever  $\dim S_f/P_i S_f \neq 0$ , it follows that  $\text{fdepth } \mathcal{F}_f \geq \text{fdepth } \mathcal{F}$ . This implies the desired conclusion.  $\square$

## 6. HILBERT SERIES OF MULTIGRADED $K$ -VECTOR SPACES

Let  $J \subset I \subset S_f$  be monomial ideals. In this section we want to show that the number of maximal Stanley spaces in any Stanley decomposition of  $I/J$  is an invariant of  $I/J$ . To prove this we introduce a new kind of Hilbert series.

Note that the localized ring  $S_f$  is naturally  $\mathbb{Z}^n$ -graded with  $\mathbb{Z}^n$ -graded components

$$(S_f)_a = \begin{cases} Kx^a, & a_i \geq 0 \text{ if } i \notin \text{supp}(f); \\ 0, & \text{otherwise.} \end{cases}$$

Let  $M = \bigoplus_{a \in \mathbb{Z}^n} M_a$  be  $\mathbb{Z}^n$ -graded  $K$ -vector space with  $\dim_K M_a < \infty$  for all  $a \in \mathbb{Z}^n$ . Then for all  $d \in \mathbb{N}$  we define

$$M_d = \bigoplus_{|a|=d} M_a \quad \text{where} \quad |a| = \sum_{j=1}^r |a_j|,$$

and set  $H(M, d) = \dim_k M_d$ . We call the function  $H(M, -): \mathbb{N} \rightarrow \mathbb{N}$  the *Hilbert function*, and the series  $H_M(t) = \sum_{d \geq 0} H(M, d)t^d$  the *Hilbert series* of  $M$ .

We consider an example to illustrate our definition. Figure 1 displays the ideal  $I = (x^3, x^2y, y^2) \subset S = K[x, y]$ . The grey area represents the monomial  $K$ -vector space spanned by the monomials in  $I$ . The slanted lines represent  $S_d = \bigoplus_{\substack{a \in \mathbb{N}^2 \\ |a|=d}} S_a$

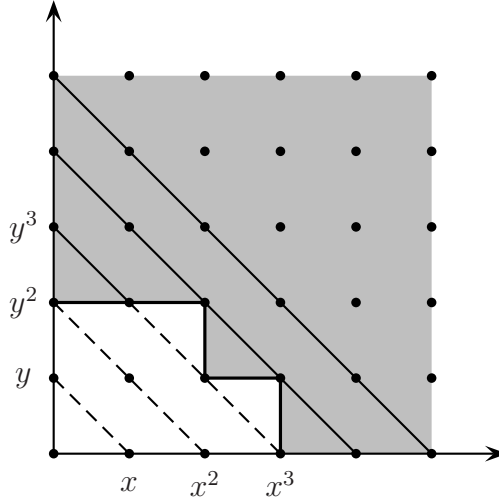


FIGURE 1.

The  $S$ -modules  $I(a)$  with  $a = (4, 3)$  is an  $S$ -submodule of  $S_f$  where  $f = xy$ . In Figure 2, the grey area displays  $I(a)$ . For all  $d \in \mathbb{N}$ , the boundaries of the squares of diagonal length  $2d$ , represent  $\bigoplus_{|a|=d} (S_f)_a$ .

In this example, if we consider  $I(a)$  as a graded  $S$ -module in the usual sense, then  $I(a)_4 = I_{11}$ , and hence  $\dim_K I(a)_4 = 12$ . If we apply our new definition of  $M_d$  to  $I(a)$ , then  $\dim_K I(a)_4 = 15$ , as can be seen in the picture.

In case that  $f = 1$ , and  $M$  is a finitely generated  $\mathbb{Z}^n$ -graded  $S$ -module with the property that the multidegree of all generators of  $M$  belong to  $\mathbb{N}^n$ , then our definition of the Hilbert function coincides with the usual definition. For properties of classical Hilbert series we refer to the book [1].

If we would define  $H(M, d)$  in the usual way as  $\dim_K(\bigoplus_{a, \sum_i a_i = d} M_a)$ , then this number would be in general infinite. On the other hand, our definition has the drawback, that the components  $(S_f)_d$  don't define a grading of  $S_f$ . In other words, we do not have in general that  $(S_f)_{d_1}(S_f)_{d_2} \subset (S_f)_{d_1+d_2}$  for all  $d_1$  and  $d_2$ . Nevertheless, we shall prove that  $H(I/J, d)$  is a polynomial function for  $d \gg 0$ .

In the case that  $J \subset I \subset S_f$  are monomial ideals, then the Hilbert function of  $I/J$  is given by

$$H(I/J, d) = |\{a \in \mathbb{Z}^n \mid x^a \in I \setminus J \text{ and } |a| = d\}|.$$

Let  $u = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S_{[n]})$ . We set  $|u| = x_1^{|a_1|} \cdots x_n^{|a_n|}$ . Then  $|u| \in \text{Mon}(S)$ .



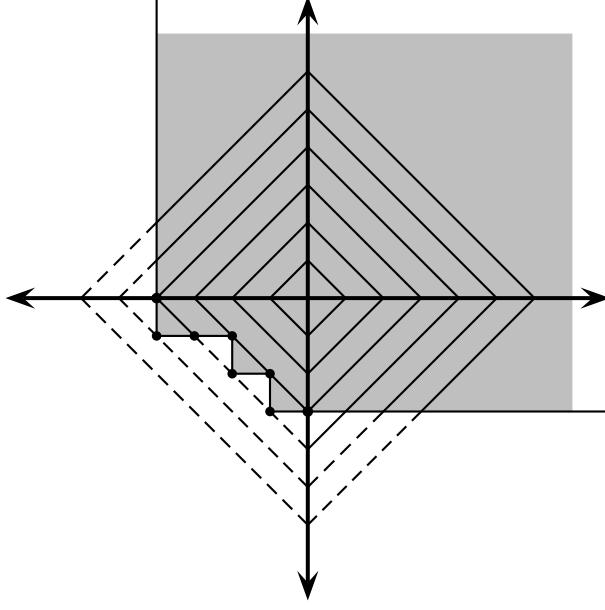


FIGURE 2.

Let  $S_{[n]} = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  and  $K[Z] \subset S_{[n]}$  be a Stanley space. We set

$$\overline{Z} = \{x_i : x_i \in Z \text{ or } x_i^{-1} \in Z\}.$$

**Lemma 6.1.** *Let  $v \in \text{Mon}(S_{[n]})$  such that*

$$\text{supp}_+(v) \cap \{i : x_i^{-1} \in Z\} = \emptyset$$

*and*

$$\text{supp}_-(v) \cap \{i : x_i \in Z\} = \emptyset.$$

*Then  $vK[Z] \cong |v|K[\overline{Z}]$ , and for each  $d \in \mathbb{N}$  we have*

$$(vK[Z])_d \simeq (|v|K[\overline{Z}])_d.$$

*Proof.* Since  $v \in \text{Mon}(S_{[n]})$  such that  $\text{supp}_+(v) \cap \{i : x_i^{-1} \in Z\} = \emptyset$  and  $\text{supp}_-(v) \cap \{i : x_i \in Z\} = \emptyset$ , it follows that  $|vw| = |v||w|$  for all  $w \in K[Z]$ . Therefore the map  $u \mapsto |u|$  induces for each  $d$  an isomorphism  $(vK[Z])_d \simeq (|v|K[\overline{Z}])_d$  of  $K$ -vector spaces.  $\square$

**Proposition 6.2.** *Let  $T_A = K[x_i^{\pm 1} : i \in A, x_{i_1}, \dots, x_{i_t}] \subset S_{[n]}$ , where  $\{i_1, \dots, i_t\} \subset [n] \setminus A$ . Let  $u \in \text{Mon}(S_{[n]})$ ,  $u = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$  such that  $a_i \geq 0$  for  $i \notin A$ . We set  $u' = \prod_{i \notin A} x_i^{a_i}$ . Then*

$$H_{uT_A}(t) = H_{u'T_A}(t) = t^{|\deg(u')|} H_{T_A}(t) = t^{|\deg(u')|} \frac{(1+t)^{|A|}}{(1-t)^d},$$

*where  $d = \dim T_A$ .*

*Proof.* Since  $uT_A = u'T_A$ , it follows that  $H_{uT_A}(t) = H_{u'T_A}(t)$ . In order to prove the second equality, we use the Stanley decomposition  $T_A = \bigoplus_{L \subset A} f_L^{-1} K[Z_L]$  where

$f_L = \prod_{i \in L} x_i$  and  $Z_L = \{x_i^{-1} : i \in L\} \cup \{x_i : i \in A \setminus L\} \cup \{x_{i_1}, \dots, x_{i_t}\}$ , from which we obtain the Stanley decomposition

$$u'T_A = \bigoplus_{L \subset A} u'f_L^{-1}K[Z_L].$$

Applying Lemma 6.1, we obtain from this decomposition the following identities

$$\begin{aligned} H_{u'T_A}(t) &= \sum_{k=0}^{|A|} \sum_{\substack{L \subset A \\ |L|=k}} H_{u'f_L^{-1}K[Z_L]}(t) = \sum_{k=0}^{|A|} \sum_{\substack{L \subset A \\ |L|=k}} H_{u'f_L K[\overline{Z_L}]}(t) \\ &= \sum_{k=0}^{|A|} \sum_{\substack{L \subset A \\ |L|=k}} \frac{t^{k+|\deg(u')|}}{(1-t)^d} = t^{|\deg(u')|} \sum_{k=0}^{|A|} \binom{|A|}{k} \frac{t^k}{(1-t)^d} \\ &= t^{|\deg(u')|} \frac{(1+t)^{|A|}}{(1-t)^d} = t^{|\deg(u')|} H_{T_A}(t), \end{aligned}$$

since  $T_A = \bigoplus_{L \subset A} f_L^{-1}K[Z_L]$  and  $H_{f_L^{-1}K[Z_L]} = \frac{t^{|L|}}{(1-t)^d}$  by Lemma 6.1, we get that

$$H_{T_A}(t) = \sum_{L \subset A} H_{f_L^{-1}K[Z_L]}(t) = \frac{(1+t)^{|A|}}{(1-t)^d}.$$

□

It follows from the next result that the Hilbert series of finitely generated  $\mathbb{Z}^n$ -graded  $S_f$ -module  $I/J$  can be written as a rational function with denominator  $(1-t)^d$ , where  $d = \dim I/J$ .

**Theorem 6.3.** *Let  $A \subset [n]$ ,  $f = \prod_{i \in A} x_i$ , and  $J \subset I \subset S_f$  be monomial ideals. Then  $I/J$  admits a filtration*

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = I/J$$

*of  $\mathbb{Z}^n$ -graded  $S_f$ -submodules of  $I/J$  such that for each  $i$  we have*

$$M_{i+1}/M_i \cong (S_f/P_i)(-a_i),$$

*where  $\{P_1, \dots, P_r\}$  is a set of  $\mathbb{Z}^n$ -graded prime ideals of  $S_f$  containing all minimal prime ideals of  $I/J$ , and where  $a_i \in \mathbb{N}^n$  with  $a_i(j) = 0$  if  $j \in A$ . Moreover,*

$$H_{I/J}(t) = \sum_{i=1}^r H_{S_f/P_i(-a_i)}(t) \text{ and } H_{S_f/P_i(-a_i)}(t) = \frac{Q_i(t)(1+t)^{|A|}}{(1-t)^{d_i}},$$

*where  $d_i = \dim S_f/P_i$  and  $Q_i(t)$  is a polynomial with  $Q_i(1) = 1$ . In particular,  $H_{I/J}(t) = Q(t)/(1-t)^d$  with  $Q(1) = k \cdot 2^{|A|}$ , where*

$$k = |\{i : \dim S_f/P_i = d\}| \quad \text{and} \quad d = \dim I/J.$$

*Proof.* Since  $J \subset I \subset S_f$  monomial ideals, we may assume that  $\text{supp}(u) \subset [n] \setminus A$  for all  $u \in G(I) \cup G(J)$ , see Proposition 1.1. Let  $S' = K[x_i : i \notin A] \subset S$  be the polynomial ring over  $K$ . Then there exist monomial ideals  $J' \subset I' \subset S'$  such that  $J = J'S_f$ ,  $I = I'S_f$ . Consider a prime filtration

$$J' = I_0 \subset I_1 \subset \dots \subset I_r = I'$$

of  $I'/J'$  where  $I_{i+1}/I_i \cong (S'/p_i)(-a_i)$ , where  $p_i$  is a monomial prime ideal  $a_i \in \mathbb{N}^n$  with  $a_i(j) = 0$  if  $j \in A$ . It follows that

$$J = J'S_f = I_0S_f \subset I_1S_f \dots \subset I_rS_f = I$$

is a prime filtration of  $I/J$  with  $I_{i+1}S_f/I_iS_f \cong S_f/P_i(-a_i)$  where  $p_iS_f = P_i$ . Because of the additivity of our Hilbert function, it suffices to show that

$$H_{S_f/P_i(-a_i)}(t) = Q_i(t)H_{S_f/P_i}(t)$$

for some polynomial  $Q_i(t)$  with  $Q_i(1) \neq 0$ . But this follows from Proposition 6.2. The rest of the statements are obvious.  $\square$

For the proof of our main result, we need the following

**Lemma 6.4.** *Let  $uK[Z]$  be a Stanley space of  $S_f$ . Then  $H_{uK[Z]}(t) = Q(t)/(1-t)^m$  where  $Q(1) = 1$  and  $m = |Z|$ .*

*Proof.* Let  $u = x_1^{a_1} \dots x_n^{a_n}$ , and let  $\mathcal{I}$  be the set of indices  $i$  for which either  $a_i > 0$  and  $x_i^{-1} \in Z$ , or  $a_i < 0$  and  $x_i \in Z$ . Then we set  $r = \sum_{i \in \mathcal{I}} |a_i|$ , and prove the lemma by induction on  $s = \min\{r, m\}$ . If  $m = 0$ , then the assertion is trivial, and if  $r = 0$ , then the result follows from Lemma 6.1.

Now let  $s > 0$ . Then we may assume that  $a_1 > 0$  and  $x_1^{-1} \in Z$ . In this case we have the following direct sum of  $K$ -vector spaces

$$uK[Z] = uK[Z \setminus \{x_1^{-1}\}] \oplus vK[Z],$$

where  $v = x_1^{a_1-1} \dots x_n^{a_n}$ . In the first summand  $m$  is smaller and in the second  $r$  is smaller than the corresponding numbers for  $uK[Z]$ . Thus we may apply our induction hypothesis according to which there exist polynomials  $Q_1(t)$  and  $Q_2(t)$  with  $Q_i(1) = 1$ , and such that

$$H_{uK[Z \setminus \{x_1^{-1}\}]}(t) = Q_1(t)/(1-t)^{m-1} \quad \text{and} \quad H_{vK[Z]}(t) = Q_2(t)/(1-t)^m.$$

It follows that  $H_{uK[Z]}(t) = Q(t)/(1-t)^m$  with  $Q(t) = Q_1(t)(1-t) + Q_2(t)$ . Since  $Q(1) = 1$ , the proof is completed.  $\square$

**Theorem 6.5.** *Let  $J \subset I \subset S_f$  be monomial ideals,  $\mathcal{D} : I/J = \bigoplus_{i=1}^r u_iK[Z_i]$  a Stanley decomposition of  $I/J$  and  $d = \max\{|Z_i| : i \in 1, 2, \dots, r\}$ . Then  $H_{I/J}(t) = P_{I/J}(t)/(1-t)^d$  with  $P_{I/J}(1) = |\{i : |Z_i| = d\}|$ .*

*Proof.* We have

$$H_{I/J}(t) = \sum_{i=1}^r H_{u_iK[Z_i]}(t)$$

By Lemma 6.4, for all  $i \in \{1, 2, \dots, r\}$  we obtain that  $H_{u_iK[Z_i]}(t) = \frac{Q_i(t)}{(1-t)^{|Z_i|}}$ , where  $Q_i(1) = 1$ . Thus

$$H_{I/J}(t) = \sum_{i=1}^r \frac{Q_i(t)}{(1-t)^{|Z_i|}} = \frac{P_{I/J}(t)}{(1-t)^d}$$

where,  $P_{I/J}(t) = \sum_{i=1}^r (1-t)^{d-|Z_i|} Q_i(t)$ . It follows that  $P_{I/J}(1) = |\{i : |Z_i| = d\}|$ .

$\square$

This Theorem implies that the number of Stanley spaces of maximal dimension in a Stanley decomposition of  $I/J$  is independent of the special Stanley decomposition of  $I/J$ .

Proposition 2.1 and Theorem 6.5 yield the following result.

**Corollary 6.6.** *The number of Stanley spaces of maximal dimension in any Stanley decomposition of  $S_f$  is equal to  $2^{|A|}$ .*

We conclude our paper with the following concrete example.

**Example 6.7.** Let  $S = K[x, y, z]$  and  $f = z$ . Then  $S_f = K[x, y, z^{\pm 1}]$ . Let  $J = (x^2) \subset I = (x, y^2) \subset S_f$  be monomial ideals in  $S_f$ . A Stanley decomposition of  $I/J$  is  $xK[y, z] \oplus xz^{-1}K[y] \oplus xz^{-2}K[y, z^{-1}] \oplus y^2K[y, z] \oplus y^2z^{-1}K[y, z^{-1}]$ . Thus in any other Stanley decomposition of  $I/J$  the number of maximal Stanley spaces is 4. Calculating the Hilbert function of  $I/J$  by using this Stanley decomposition we find that  $H_{I/J}(t) = \frac{t+2t^2+t^3}{(1-t)^2}$ . Thus  $P(t) = t + 2t^2 + t^3$ , and  $P(1) = 4$ , as expected.

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